Math 564: Real analysis and measure theory
Lecture 16

Prop. If a measure space $(X, \mathcal{B}, \mu)$ is ctbly generated (i.e. Meas_μ is separable) then there is a ctbl collection of simple functions which is dense in $L^1(X,\mu)$ in the L^1-metric. In particular, $L^1(X,\mu)$ is separable.

Examples. (a) $L^1(\mathbb{R}^d, \lambda)$ is separable because $\mathbb{R}^d$ is 2^{nd} ctbl and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$.

(b) $L^1(A^{\mathbb{N}}, \mu)$, where A is nonempty finite and μ is a Bernoulli measure, is separable because $A^{\mathbb{N}}$ is 2^{nd} ctbl and $\mathcal{B} = \mathcal{B}(A^{\mathbb{N}})$.

(c) $\ell^1(X)$, for some set X, is separable if and only if X is ctbl.
Indeed, if X is ctbl then $\mathcal{B} = \mathcal{P}(X)$ is generated by the singletons.
If X is unctbl, then the family $\{\mathbb{1}_{\{x\}}\}_{x\in X}$ is discrete and unctbl, hence $\ell^1(X)$ is not separable.

(d) For any σ-finite Borel measure μ on a 2^{nd} ctbl metric space X has a separable $L^1(X,\mu)$.

We now discuss the completeness of $L^1(X,\mu)$, for which we first give a criterion of completeness for normed vector spaces.

Def. Let $(V, \|\cdot\|)$ be a (pseudo) normed vector space, viewed as a metric space with metric $d(f,g) := \|f-g\|$ for all $f,g\in V$. For a series $\sum_{n\in\mathbb{N}} f_n$ of elements of V, we say that it

- **converges in norm** if there is $f\in V$ such that $\sum_{n\le N} f_n \to f$ as $N\to\infty$ in norm, i.e. $\|f - \sum_{n\le N} f_n\| \to 0$ as $N\to\infty$. In this case, we simply write $\sum_{n\in\mathbb{N}} f_n = f$.
- **absolutely converges** if $\sum_{n\in\mathbb{N}} \|f_n\| < \infty$.

Criterion for completeness. Let $(V, \|\cdot\|)$ be a (pseudo) normed vector space. Then V is complete (every Cauchy sequence converges) $\iff$ every absolutely convergent series converges in norm.

Proof. $\Rightarrow$. Suppose V is complete and let $\sum_{n\in\mathbb{N}} f_n$ be an absolutely convergent series. But then the sequence $g_N := \sum_{n\leq N} f_n$ is Cauchy: for $M \geq N$,

$$\|g_N - g_M\| = \|\sum_{n=N+1}^{M} f_n\| \leq \sum_{n=N+1}^{M} \|f_n\| \leq \sum_{n>N} \|f_n\| \to 0$$ as $N\to\infty$ because the tail of a convergent series converges to 0. Hence (g_N) has a limit f so $\sum_{n\in\mathbb{N}} f_n = f$.

$\Leftarrow$. Let (f_n) be a Cauchy sequence. Then we consider the series $\sum_{n\in\mathbb{N}}(f_{n+1}-f_n)$ because it partial sums $\sum_{n<N}(f_{n+1}-f_n) = f_N - f_0$ converge iff (f_n) converges. Thus it is enough to show that this series converges absolutely, i.e. $\sum_{n\in\mathbb{N}} \|f_{n+1}-f_n\| < \infty$. But this may not be true (e.g. $f_n := \frac{1}{\sqrt{n}}$ in $V := \mathbb{R}$), so we use the accelleration trick for Cauchy sequences, i.e. we recall that a Cauchy sequence converges iff it has a convergent subsequence, and we pass to a subsequence (f_{n_k}) such that $\|f_{n_{k+1}} - f_{n_k}\| \leq 2^{-k}$ (which we can do using the Cauchy property). So WLOG, we assume that (f_n) was like this to begin with, i.e. $\|f_{n+1}-f_n\| \leq 2^{-n}$. Then the series $\sum_{n\in\mathbb{N}}(f_{n+1}-f_n)$ converges absolutely: $\sum_{n\in\mathbb{N}}\|f_{n+1}-f_n\| \leq \sum_{n\in\mathbb{N}} 2^{-n} < \infty$, hence it converges in norm, therefore so does the sequence (f_n). $\square$

Theorem. For any measure space $(X, \mathcal{B}, \mu)$, the space $L^1(X, \mathcal{B}, \mu)$ is complete.

Proof. Let $\sum_{n\in\mathbb{N}} f_n$ be an absolutely convergent series, i.e. $\sum_{n\in\mathbb{N}} \|f_n\|_1 < \infty$. Then $g := \sum_{n\in\mathbb{N}} |f_n|$ dominates the sequence of partial sums $\sum_{n<N} f_n$ and g is in $L^1(X,\mu)$ since $\|g\|_1 = \int \sum_{n\in\mathbb{N}} |f_n|\, d\mu \overset{MCT}{=} \sum_{n\in\mathbb{N}} \int |f_n|\, d\mu = \sum_{n\in\mathbb{N}} \|f_n\| < \infty$. In particular, the function $\sum_{n\in\mathbb{N}} |f_n|$ is finite a.e. hence

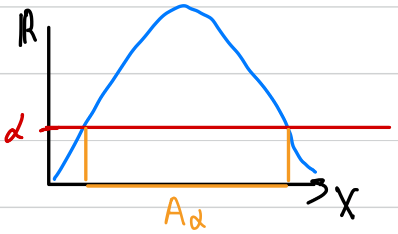
the series $\sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely (i.e. $\sum_{n\in\mathbb{N}} |f_n(x)| < \infty$) for a.e. $x\in X$, hence converges a.e., and let $f(x) := \sum_{n\in\mathbb{N}} f_n(x)$ a.e.. Then f is measurable being the limit of partial sums (hence measurable functions) and $|f| = |\sum_{n\in\mathbb{N}} f_n| \leq \sum_{n\in\mathbb{N}} |f_n| = g$. Thus we may now apply DCT to the sequence $\left(\sum_{n<N} f_n\right)_{N\in\mathbb{N}}$ of partial sums and get that these partial sums converge to f in L^1-metric. $\square$

Properties of integrable functions.

Chebyshev's inequality. For each $f\in L^1(X,\mu)$ and $\alpha\in(0,\infty]$, we have

$$\mu(\underbrace{\{x\in X : |f(x)|>\alpha\}}_{A_\alpha}) \le \frac{1}{\alpha}\cdot\|f\|_1.$$

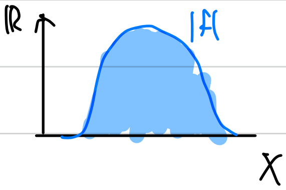
Proof.



$$\|f\|_1 = \int |f|\,d\mu \ge \int_{A_\alpha} |f|\,d\mu \ge \int_{A_\alpha} \alpha\,d\mu = \alpha\cdot\int_{A_\alpha} d\mu = \alpha\cdot\mu(A_\alpha). \quad\square$$

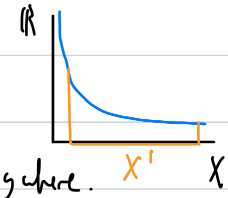
Cor. For each $f\in L^1(X,\mu)$, its support $\mathrm{supp}(f) := \{x\in X : f(x)\neq 0\}$ is σ-finite for μ, i.e. $\mathrm{supp}(f) = \bigcup_{n\in\mathbb{N}} B_n$, where B_n is measurable and $\mu(B_n)<\infty$.

Proof. Let $B_n := \{x\in X : |f(x)|>\frac{1}{n}\}$, $n\in\mathbb{N}^+$. Then $\mathrm{supp}(f) = \bigcup_{n\in\mathbb{N}} B_n$ and $\mu(B_n)\le n\cdot\|f\|_1<\infty$ by Chebyshev. $\square$



Recall that for any $f\in L^1(X,\mu)$, we have a measure $\mu_{|f|}(A) := \int_A |f|\,d\mu$.

99% boundedness. For any $f\in L^1(X,\mu)$ and $\varepsilon>0$, there is a meas. $X'\subseteq X$ such $f|_{X'}$ is bdd and $\mu_{|f|}(X\setminus X') = \int_{X\setminus X'} |f|\,d\mu < \varepsilon$.



Proof. Let $X_n := \{x\in X : |f(x)|\le n\}$. WLOG, change f so $|f|<\infty$ everywhere. Then $X = \bigcup_{n\in\mathbb{N}} X_n$ so by increasing monotone converge for $\mu_{|f|}$, we have

$$\lim_{n\to\infty} \mu_{|f|}(X_n) = \mu_{|f|}(X) < \infty,$$ so $\mu_{|f|}(X\setminus X_n)\to 0$ as $n\to\infty$, so for large enough $n\in\mathbb{N}$,

$\mu(X\setminus X_n) < \varepsilon$, hence take $X' := X_n$. $\square$

Def. Let $(X, \mathcal{B})$ be a measurable space and μ, ν be measures on $(X, \mathcal{B})$. We say that ν is **absolutely continuous** wrt μ, and write $\nu \ll \mu$, if every μ-null set is ν-null.

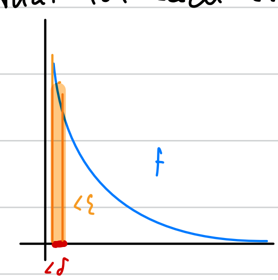
Example. For any $f \in L^1(X, \mu)$, the measure $\mu_{|f|}$ is finite and $\mu_{|f|} \ll \mu$ because if $B \subseteq X$ is μ-null then $\mu_{|f|}(B) = \int_B |f| \, d\mu = 0$.

The following justifies the terminology of absolutely continuous:

Prop. Let ν, μ be measures on a measurable space $(X, \mathcal{B})$. If ν is finite, then
$\nu \ll \mu$ iff $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $\mu(B) < \delta \Rightarrow \nu(B) < \varepsilon$ for all $B \in \mathcal{B}$.

Proof. $\Leftarrow$. Trivial because $0 < \varepsilon$.
$\Rightarrow$. We prove the contrapositive. Assume $\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists B_\delta \in \mathcal{B}$ $\mu(B_\delta) < \delta$ but $\nu(B_\delta) \geq \varepsilon$.
By the first application of Borel–Cantelli (at the end of Lecture 7) applied to the collection
$$\mathcal{C} := \{ B \in \mathcal{B} : \nu(B) \geq \varepsilon \},$$
and μ, this collection admits a μ-almost vanishing sequence, i.e. decreasing $(B_n) \subseteq \mathcal{C}$ with $\mu(\bigcap_{n \in \mathbb{N}} B_n) = 0$. But because ν is a finite measure, decreasing monotone convergence applies to (B_n) and ν yielding $\nu(\bigcap_{n \in \mathbb{N}} B_n) = \lim_{n \to \infty} \nu(B_n) \geq \varepsilon > 0$, so $\nu \not\ll \mu$. $\square$

Cor (absolute continuity of integrable functions). For any $f \in L^1(X, \mu)$, we have that for each $\varepsilon > 0 \; \exists \delta > 0$ such that whenever $\mu(B) < \delta$, we have $\int_B |f| \, d\mu < \varepsilon$.



Remark. This property (abs. cont. for integrable functions) also follows directly from 99% boundedness (HW).